Problem Sheet II

Physical Cosmology
Part III Mathematical Tripos
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[1] Using the relation between apparent magnitude m, absolute magnitude M and the luminosity distance $d_{\rm L}$:

$$m - M = 5 \log \left(\frac{d_{\rm L}}{10 \text{ pc}} \right) ,$$

estimate the luminosity distance $d_{\rm L}$ to which the following objects could be seen if $m = 20$: (i) a star like sun ($M = 4.72$); (ii) a globular cluster; (iii) a bright galaxy.

[2] Derive the Hubble parameter $H(z) = \dot{a}/a$ for a universe with an equation of state $p = w\rho$, where w is a constant.
Using $H(z)$, derive the luminosity distance, the angular distance, the volume element and the age of the Universe in this case.

[3] (a) *Reconstruction of the potential of dark energy*
Assume a flat universe, where the coordinate distance is given by $r(z) = \int_0^z dx/H(x)$, and H is the Hubble parameter. Derive an expression for H^2 in terms of dr/dz, and $\ddot{a}/a$ in terms of dr/dz and d^2r/dz^2.
Assume that the dark energy is given by a scalar field rolling down a potential $V(\phi)$, and that the energy density and pressure of a scalar field are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

and

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) .$$

Use the Friedmann equations with a scalar field dark energy and matter component (with present-epoch $\Omega_{\mathrm{m},0}$) to show that the following parametric solution exists:

$$V[\phi(z)] = \frac{1}{8\pi G}\left[\frac{3}{(dr/dz)^2} + (1+z)\frac{d^2r/dz^2}{(dr/dz)^3}\right] - \frac{3\Omega_{\mathrm{m},0}H_0^2(1+z)^3}{16\pi G} \quad (1)$$

$$\frac{d\phi}{dz} = \mp\frac{dr/dz}{1+z}\left[-\frac{1}{4\pi G}\frac{(1+z)d^2r/dz^2}{(dr/dz)^3} - \frac{3\Omega_{\mathrm{m},0}H_0^2(1+z)^3}{8\pi G}\right]^{1/2}. \quad (2)$$

Discuss reasons why it might be tricky to apply these equations to observations.

(b) *Reconstruction of the equation of state parameter*

Assume that the dark energy component is characterized by the equation of state parameter $w(z) = p_{\mathrm{de}}/\rho_{\mathrm{de}}$ and the present-epoch density parameter $\Omega_{\mathrm{de},0}$. Show that

$$\rho_{\mathrm{de}}(z) = \frac{3H_0^2}{8\pi G}\Omega_{\mathrm{de},0}\exp\left[3\int_0^z [1+w(z)]\,d\ln(1+z)\right].$$

Show that:

$$1 + w(z) = \frac{1+z}{3}\frac{3H_0^2\Omega_{\mathrm{m},0}(1+z)^2 + 2(d^2r/dz^2)/(dr/dz)^3}{H_0^2\Omega_{\mathrm{m},0}(1+z)^3 - (dr/dz)^{-3}}.$$

Discuss qualitatively what astrophysical quantities and observations could be used for the reconstruction.